UNIQUENESS OF HIGHER GAUDIN HAMILTONIANS

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Abstract. For any semisimple Lie algebra $\mathfrak{g}$, the universal enveloping algebra of the infinite-dimensional pro-nilpotent Lie algebra $\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ contains a large commutative subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$. This subalgebra comes from the center of the universal enveloping of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level and gives rise to the construction of higher hamiltonians of the Gaudin model (due to Feigin, Frenkel and Reshetikhin). Though there are no explicit formulas for the generators of $\mathcal{A}$ known in general, the "classical analogue" of this subalgebra, i.e. the associated graded subalgebra in the Poisson algebra $A \subset S(\mathfrak{g}_-)$, can be easily described. In this note we show that the "classical" subalgebra $A \subset S(\mathfrak{g}_-)$ is the Poisson centralizer of some of its quadratic elements, and deduce from this that the "quantum" subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ is uniquely determined by the space of quadratic elements of the classical one. In particular, this means that some different constructions of higher Gaudin hamiltonians (namely, Feigin-Frenkel-Reshetikhin's method and Talalaev-Chervov's method), give the same family of commuting operators. The proof uses some ideas of the previous paper math.QA/0608586.

1. Introduction.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We consider the infinite-dimensional pro-nilpotent Lie algebra $\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ – it is a "half" of the corresponding affine Kac–Moody algebra $\hat{\mathfrak{g}}$. The universal enveloping algebra $U(\mathfrak{g}_-)$ bears a natural filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(\mathfrak{g}_-)$ by the Poincaré–Birkhoff–Witt theorem. The commutator operation on $U(\mathfrak{g}_-)$ defines the Poisson–Lie bracket $\{\cdot,\cdot\}$ on $S(\mathfrak{g}_-)$: for the generators $x, y \in \mathfrak{g}_-$ we have $\{x, y\} = [x, y]$. For any $g \in \mathfrak{g}$, we denote the element $g \otimes t^{-m} \in \mathfrak{g}_-$ by $g[-m]$.

The Poisson algebra $S(\mathfrak{g}_-)$ contains a large Poisson-commutative subalgebra $A \subset S(\mathfrak{g}_-)$. This subalgebra can be constructed as follows.

Consider the following derivation of the Lie algebra $\mathfrak{g}_-$:

$$\partial_t(g[-m]) = -mg[-m-1] \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots \tag{1}$$

The derivation (1) extends to the derivation of the associative algebras $S(\mathfrak{g}_-)$ and $U(\mathfrak{g}_-)$.

Let $i_{-1} : S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$ be the embedding, which maps $g \in \mathfrak{g}$ to $g[-1]$. Let $\Phi_k,\ k = 1, \dots, \mathrm{rk}\,\mathfrak{g}$ be the generators of the algebra of invariants $S(\mathfrak{g})^{\mathfrak{g}}$.

Fact 1. [BD, FFR, Fr2]

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(1) *The subalgebra* $A\subset S(\mathfrak{g}_-)$ *generated by the elements* $\partial_t^n\overline{S_k}$, $k=1,\dots,\operatorname{rk}\mathfrak{g}$, $n=0,1,2,\dots$, *where* $\overline{S_k}=i_{-1}(\Phi_k)$, *is Poisson-commutative.*
(2) *There exist the elements* $S_k\in U(\mathfrak{g}_-)^{\mathfrak{g}}$ *such that* $\operatorname{gr}S_k=\overline{S_k}$ *and the subalgebra* $\mathcal{A}\subset U(\mathfrak{g}_-)$ *generated by* $\partial_t^n S_k$, $k=1,\dots,\operatorname{rk}\mathfrak{g}$, $n=0,1,2,\dots$ *is commutative.*

Remark. The generators of the subalgebra $A\subset S(\mathfrak{g}_-)$ can be described in the following equivalent way. Let $i(z):S(\mathfrak{g})\hookrightarrow S(\mathfrak{g}_-)$ be be the embedding depending on the formal parameter z, which maps $g\in\mathfrak{g}$ to $\sum\limits_{k=1}^{\infty}z^{k-1}g[-k]$. Then the coefficients of the power series $\overline{S_k}(z)=i(z)(\Phi_k)$ in z freely generate the subalgebra $A\subset S(\mathfrak{g}_-)$.

Remark. The subalgebra $\mathcal{A}\subset U(\mathfrak{g}_-)$ comes from the center of $U(\hat{\mathfrak{g}})$ at the critical level by the AKS-scheme (see [FFR, ER, ChT]).

Remark. No general explicit formulas for the elements S_k are known in general. For the quadratic Casimir element Φ_1, the corresponding element $S_1\in\mathcal{A}$ is obtained from $\overline{S_1}=i_{-1}(\Phi_1)$ by the symmetrization map. More explicitly, we have $S_1=\sum\limits_{a=1}^{\dim\mathfrak{g}}x_a[-1]^2$, where $\{x_a\}$, $a=1,\dots,\dim\mathfrak{g}$ is an orthonormal (with respect to the Killing form) basis of $\mathfrak{g}$.

Remark. For $\mathfrak{g}=sl_r$ explicit formulas for S_k were obtained by Talalaev and Chervov in [ChT]. We shall reproduce these formulas in section 2.

The main result of the present paper is the following

Theorem 1. (1) *The subalgebra* $A\subset S(\mathfrak{g}_-)$ *is a Poisson centralizer of the element* $\overline{S_1}=i_{-1}(\Phi_1)\in A$ *(where* Φ_1 *is the quadratic Casimir). In particular,* $A\subset S(\mathfrak{g}_-)$ *is a* maximal *Poisson-commutative subalgebra in* $S(\mathfrak{g}_-)$.
(2) *The subalgebra* $\mathcal{A}\subset U(\mathfrak{g}_-)$ *is a centralizer of the element* $S_1\in\mathcal{A}$.
(3) *Any commutative subalgebra* $\tilde{\mathcal{A}}\subset U(\mathfrak{g}_-)^{\mathfrak{g}}$, *such that* $\operatorname{gr}\tilde{\mathcal{A}}=A$, *coincides with* $\mathcal{A}\subset U(\mathfrak{g}_-)^{\mathfrak{g}}$ *(in other words, there is a unique* G*-invariant lifting of the commutative subalgebra* $A\subset S(\mathfrak{g}_-)$ *to the universal enveloping algebra).*

Remark. In [R] a similar fact for Mischenko–Fomenko subalgebras of the Poisson algebra $S(\mathfrak{g})$ is proved. Unfortunately, the ideas of [R] can not be generalized straightforwardly to our situation. The proof of Theorem 1 uses also some new ideas.

We will prove the Theorem in section 3. Theorem 1 has some applications in the quantization of higher hamiltonians of the Gaudin model. We discuss this in section 2.

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2. Gaudin model

Gaudin model was introduced in [G1] as a spin model related to the Lie algebra sl_2, and generalized to the case of an arbitrary semisimple Lie algebra in [G],

13.2.2. The generalized Gaudin model has the following algebraic interpretation. For any $x \in \mathfrak{g}$, set $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes n}$ (x stands at the ith place). Let $\{x_a\}$, $a = 1, \ldots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to Killing form, and let $z_1, \ldots, z_n$ be pairwise distinct complex numbers. The hamiltonians of Gaudin model are the following commuting elements of $U(\mathfrak{g})^{\otimes n}$:

$$H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}. \tag{2}$$

In [FFR], a large commutative subalgebra $\mathcal{A}(z_1, \ldots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ containing H_i was constructed. Generators of this algebra are known as higher Gaudin hamiltonians. The commutative subalgebra $\mathcal{A}(z_1, \ldots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ is the image of the subalgebra $\mathcal{A} \subset U(\mathfrak{g}_-)$ under the homomorphism $U(\mathfrak{g}_-) \to U(\mathfrak{g})^{\otimes n}$ of specialization at the points $z_1, \ldots, z_n$ (see [FFR, ER]).

In [Tal] D. Talalaev constructed explicitly some elements of $U(\mathfrak{g})^{\otimes n}$ commuting with quadratic Gaudin hamiltonians for the case $\mathfrak{g} = \mathfrak{gl}_r$. The formulas of [Tal] are universal, i.e. actually they describe a commutative subalgebra of $U(\mathfrak{g}_-)$ which gives a commutative subalgebra of $U(\mathfrak{g})^{\otimes n}$ as the image of the specialization homomorphism at the points $z_1, \ldots, z_n$ (see [ChT]).

Namely, set

$$L(z) = \sum_{1 \leq i,j \leq r} \sum_{n=1}^{\infty} z^{n-1} e_{ij}[-n] \otimes e_{ji} \in U(\mathfrak{g}_-) \otimes \operatorname{End} \mathbb{C}^r,$$

where z is a formal parameter, and consider the following differential operator in z with the coefficients from $U(\mathfrak{g}_-)$:

$$D = \operatorname{Tr} A_r \prod_{i=1}^{r} (L(z)^{(i)} - \partial_z) = \partial_z^r + \sum_{k=1}^{r} \sum_{n=1}^{\infty} Q_{n,k} z^{k-1} \partial_z^{r-k}.$$

Here we denote by $L(z)^{(i)} \in U(\mathfrak{g}_-) \otimes (\operatorname{End} \mathbb{C}^r)^{\otimes r}$ the element obtained by putting $L(z)$ to the i-th tensor factor, and A_r denotes the projector onto $U(\mathfrak{g}_-) \otimes \operatorname{End}(\Lambda^r \mathbb{C}^r) \subset U(\mathfrak{g}_-) \otimes (\operatorname{End} \mathbb{C}^r)^{\otimes r}$. It follows from the works [ChT, Tal] by Chervov and Talalaev that the elements $Q_{n,k} \in U(\mathfrak{g}_-)$ pairwise commute.

It is easy to see the Poisson-commutative subalgebra in $S(\mathfrak{g}_-)$ generated by $\operatorname{gr} Q_{n,k} \in S(\mathfrak{g}_-)$ coincides with A, i.e. $\operatorname{gr} \sum\limits_{n=1}^{\infty} Q_{n,k} z^{k-1} = \overline{S_k}(z) = i(z)(\Phi_k) \in S(\mathfrak{g}_-)$, where we take as the generators Φ_k of the algebra of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ the coefficients of the characteristic polynomial (see Remark 1 in [ChT]).

From Theorem 1 we obtain the following

Corollary 1. *In the case of $\mathfrak{g} = \mathfrak{gl}_r$, the higher Gaudin hamiltonians in the sense of* [Tal] *coincide with higher Gaudin hamiltonians in the sense of* [FFR].

3. Proof of Theorem 1.

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let e, h, f be a principal $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{g}$ (here $h \in \mathfrak{h}$). Let $\mathfrak{z}_{\mathfrak{g}}(f)$ be the centralizer of f in $\mathfrak{g}$. Now, we will prove some analogs of the following classical results for the subalgebra $A \subset S(\mathfrak{g}_-)$.

Fact 2. (Kostant [Ko]) *The homomorphism of the restriction to the affine subspace* $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[e + \mathfrak{z}_{\mathfrak{g}}(f)]$ *induces an algebra isomorphism* $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \tilde{\to} \mathbb{C}[e + \mathfrak{z}_{\mathfrak{g}}(f)]$.

Fact 3. (Chevalley, see e.g. [He]) *The image of* $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ *under the restriction to the Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is the algebra of invariants of the Weyl group action on* $\mathfrak{h}$. *In particular* $\mathbb{C}[\mathfrak{h}]$ *is an algebraic extension of the image of* $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$.

Let $V \subset \mathfrak{g}$ be an h-invariant complement of the subspace $\mathfrak{z}_{\mathfrak{g}}(f) \subset \mathfrak{g}$. We consider the homomorphism $\pi : S(\mathfrak{g}_-) \to S(\mathfrak{z}_{\mathfrak{g}}(f)_-)$ defined as follows.

$$\pi(x[-m]) = x[-m] \quad \forall x \in \mathfrak{z}_{\mathfrak{g}}(f),$$

and

$$\pi(x[-m]) = \delta_{1m}\langle x, f\rangle \quad \forall x \in V.$$

Let $\psi : \mathfrak{g}_- \to \mathfrak{h}_-$ be an $\mathfrak{h}$-invariant projection. This projection extends to the algebra homomorphism $S(\mathfrak{g}_-) \to S(\mathfrak{h}_-)$, which we denote by the same letter ψ.

Lemma 1. (1) *The homomorphism* $\pi : A \to S(\mathfrak{z}_{\mathfrak{g}}(f)_-)$ *is an isomorphism of algebras.*
(2) $\psi(A) \subset S(\mathfrak{h}_-)$ *is an algebraic extension.*

Proof. Let us prove the first assertion. By Fact 2 the restriction of the homomorphism π to the subalgebra $\mathbb{C}[\overline{S_1}, \dots, \overline{S_l}] \subset A$ induces an algebra isomorphism $\pi : \mathbb{C}[\overline{S_1}, \dots, \overline{S_l}] \to S(\mathfrak{z}_{\mathfrak{g}}(f)[-1])$. Now, it remains to notice that the homomorphism $\pi : S(\mathfrak{g}_-) \to S(\mathfrak{z}_{\mathfrak{g}}(f)_-)$ commutes with the operator ∂_t and therefore

$$\pi : A = \mathbb{C}[\overline{S_1}, \dots, \overline{S_l}, \dots, \partial_t^k \overline{S_1}, \dots, \partial_t^k \overline{S_l}, \dots] \to S(\mathfrak{z}_{\mathfrak{g}}(f)_-)$$

is an algebra isomorphism as well.

Now, let us prove the second assertion of the Lemma. By Fact 3 we have that every element of the subspace $\mathfrak{h}[-1] \subset S(\mathfrak{h}_-)$ is algebraic over $\psi(A)$. Since the homomorphism $\psi : S(\mathfrak{g}_-) \to S(\mathfrak{h}_-)$ commutes with the operator ∂_t and the subalgebra $A \subset S(\mathfrak{g}_-)$ is stable under ∂_t, we see that for any positive integer k every element of the subspace $\partial_t^k(\mathfrak{h}[-1]) = \mathfrak{h}[-k]$ is algebraic over $\psi(A)$ as well. Thus, $\psi(A) \subset S(\mathfrak{h}_-) = S(\bigoplus_{k=1}^{\infty} \mathfrak{h}[-k])$ is an algebraic extension. $\square$

Lemma 2. *The subalgebra* A *is algebraically closed in* $S(\mathfrak{g}_-)$.

Proof. Assume that there is an element $a \in S(\mathfrak{g}_-)$ such that $a \notin A \quad p_N a^N + p_{N-1}a^{N-1} + \dots + p_0 = 0$ for some $p_0, \dots, p_N \in A$. Suppose that the number N is minimal possible. By the first assertion of Lemma 1 without loss of generality we can assume that $\pi(a) = 0$ without loss of generality. This means, in particular, that $\pi(p_0) = 0$, and hence, by the first assertion of Lemma 1 $p_0 = 0$. Since $S(\mathfrak{g}_-)$ has no divisors of zero, we have $p_N a^{N-1} + p_{N-1}a^{N-2} + \dots + p_1 = 0$. This means that N is not minimal. $\square$

Now, we define a 1-parametric family of automorphisms φ_s of the Poisson algebra $S(\mathfrak{g}_-)$: for $x \in \mathfrak{g}$ set $\varphi_s(x[-k]) = x[-k] + s\delta_{1,k}\langle h, x\rangle$ (here $\langle \cdot, \cdot \rangle$ denotes the Killing form). The following assertion is checked by direct computation.

Lemma 3. $\lim\limits_{s \longrightarrow \infty} \frac{\overline{S_1} - s^2\Phi_1(h)}{2s} = h[-1]$.

Lemma 4. *$S(\mathfrak{h}_-)$ is the centralizer of $h[-1]$ in $S(\mathfrak{g}_-)$.*

Proof. Fix any ordering on the set of roots of $\mathfrak{g}$ and define an ordering on the generators of $S(\mathfrak{g}_-)$ in the following way:

$$h_{\alpha_i}[-k] < e_\alpha[-m]\,\forall k, m, \alpha_i, \alpha;$$

$$e_\alpha[-k] < e_\beta[-m]\,\forall k, m, \text{ if}\,\alpha < \beta;$$

$$h_{\alpha_i}[-k] < h_{\alpha_j}[-m]\,\forall k, m, \text{ if}\,\alpha_i < \alpha_j;$$

$$x[-k] < x[-m], \text{ if}\,k < m;$$

Assume that there exists $f \in S(\mathfrak{g}_-)$ commuting with $h[-1]$ such that $f \notin S(\mathfrak{h}_-)$. Let

$$\prod_{i=1}^{l}\prod_{k=1}^{\infty}(h_{\alpha_i}[-k])^{m_{i,k}}\prod_{\alpha\in\Delta}\prod_{k=1}^{\infty}(e_\alpha[-k])^{n_{\alpha,k}}$$

be the leading monomial of f with respect to our ordering, and let α_{max} be the maximal root such that $n_{\alpha,k} \neq 0$ for some k. Let k_{max} be the largest of such k (by the assumption $f \notin S(\mathfrak{h}_-)$, therefore such α_{max} and k_{max} exist). Then the leading monomial of the commutator $\{h[-1], f\}$ is

$$n_{\alpha_{max},k_{max}}\langle h, \alpha_{max}\rangle\frac{e_{\alpha_{max}}[-k_{max}-1]}{e_{\alpha_{max}}[-k_{max}]}\prod_{i=1}^{l}\prod_{k=1}^{\infty}(h_{\alpha_i}[-k])^{m_{i,k}}\prod_{\alpha\in\Delta}\prod_{k=1}^{\infty}(e_\alpha[-k])^{n_{\alpha,k}} \neq 0.$$

This contradicts our assumption. The Lemma is proved. $\square$

Proof of Theorem 1. (1). Let $I \subset S(\mathfrak{g}_-)$ be the kernel of the homomorphism $\psi : S(\mathfrak{g}_-) \to S(\mathfrak{h}_-)$, i.e. the ideal generated by all $e_\alpha[-m]$, $\alpha \in \Delta$, $m = 1, 2, \dots$. Note that all the automorphisms φ_s preserve the ideal I. Assume that there is an element a in the Poisson centralizer of $\overline{S_1}$ in $S(\mathfrak{g}_-)$, which do not belong to A. By Lemma 2 the element a is transcendental over A. By the second assertion of Lemma 1, we can pass to a polynomial expression in a with the coefficients from A to make $\psi(a) = 0$, i.e. $a \in I$. For appropriate k, there exist a non-zero limit $\overline{a} := \lim\limits_{s\longrightarrow\infty}\frac{\varphi_s(a)}{s^k} \in I$. Moreover, by Lemma 3 $\overline{a}$ belongs to the centralizer of $h[-1]$ in $S(\mathfrak{g}_-)$. On the other hand, $\overline{a} \in I$, an hence $\overline{a} \notin S(\mathfrak{h}_-)$. This contradicts Lemma 4. Thus, the Poisson centralizer of $\overline{S_1}$ in $S(\mathfrak{g}_-)$ coincides with A.

(2). Let Z be the centralizer of S_1 in the associative algebra $U(\mathfrak{g})_-$. We have $\operatorname{gr} Z \subset A$ and $\mathcal{A} \subset Z$, hence $Z = \mathcal{A}$.

(3). It is clear that S_1 is the unique – up to an additive constant – lifting of $\overline{S_1}$ to $U(\mathfrak{g}_-)^{\mathfrak{g}}$. Since the centralizer does not depend on adding a constant, we see that $\mathcal{A}$ is the unique lifting of the Poisson commutative subalgebra $A \subset S(\mathfrak{g}_-)$ to the associative algebra $U(\mathfrak{g}_-)^{\mathfrak{g}}$. Thus, Theorem 1 is proved. $\square$

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